Решения на задачи за изпит по ОТО

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DATE

1. Решение Собствения часовник в собствената координатна система винаги върви най-бързо, защото отнема време на светлината от единия брат да достигне другия брат. Всеки брат изминава еднакво разстояние с еднаква скорост, значи за всеки брат е изминало еднакво количество време.

2. Решение Смятало се е, че светлината се движи през някаква невидима среда (наречена "етер"). Ако етерът съществувал, то Земята ще се движи през него и би пораждала етеров вятър. Светлинен лъч ще се движи по-бързо по посоката на вятъра, по-бавно срещу и нещо средно перпендикулярно.

Разглеждайки два перпендикулярни снопа, изминаващи еднакво разстояние от еднакъв източник, стигайки еднаква дестинация, Майкелсон и Морли експериментално намират обратното: скоростта на светлината никога не се променя (и етер не съществува).

3. Решение $00800028\ mod\ 3 = 0$, тоест ще докажем, че ковариантната производна на тензор от тип $(0,2)$ е тензор от тип $(0,3)$.

$$\nabla g = \nabla_k\, g_{ab}\, \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}x^k = \left(\frac{\partial g_{ab}}{\partial x^k} - \Gamma^m{}_{ak}\, g_{mb} - \Gamma^m{}_{kb}\, g_{ma} \right) \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}x^k$$

Правим смяна на променлива, от x към y, което означава че:

$$\tilde{\Gamma}^i{}_{jk} = \Gamma^a{}_{bc} \frac{\partial y^i}{\partial x^a} \frac{\partial x^b}{\partial y^j} \frac{\partial x^c}{\partial y^k} + \frac{\partial y^i}{\partial x^p} \frac{\partial^2 x^p}{\partial y^j \partial y^k} \implies \Gamma^a{}_{bc} = \left(\tilde{\Gamma}^i{}_{jk} - \frac{\partial y^i}{\partial x^p} \frac{\partial^2 x^p}{\partial y^j \partial y^k} \right) \frac{\partial x^a}{\partial y^i} \frac{\partial y^j}{\partial x^b} \frac{\partial y^k}{\partial x^c} \qquad \mathrm{d}x^a = \frac{\partial x^a}{\partial y^i} \mathrm{d}y^i$$

$$g = g_{ab} \mathrm{d}x^a \mathrm{d}x^b = g_{ab} \frac{\partial x^a}{\partial y^i} \mathrm{d}y^i \frac{\partial x^b}{\partial y^j} \mathrm{d}y^j = g_{ab} \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \mathrm{d}y^i \mathrm{d}y^j = \tilde{g}_{ij} \mathrm{d}y^i \mathrm{d}y^j \implies g_{ab} = \tilde{g}_{ij} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b}$$

Заедно с $\dfrac{\partial}{\partial x^i} = \dfrac{\partial}{\partial y^a} \dfrac{\partial y^a}{\partial x^i}$ следва че:

$$\frac{\partial g_{ab}}{\partial x^k} = \frac{\partial y^l}{\partial x^k} \frac{\partial}{\partial y^l} \left(\tilde{g}_{ij} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} \right) = \frac{\partial y^l}{\partial x^k} \left(\tilde{g}_{ij} \frac{\partial}{\partial y^l} \left(\frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} \right) + \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} \frac{\partial}{\partial y^l} \left(\tilde{g}_{ij} \right) \right)$$

$$= \frac{\partial y^l}{\partial x^k} \left(\tilde{g}_{ij} \left(\frac{\partial y^i}{\partial x^a} \frac{\partial}{\partial y^l} \left(\frac{\partial y^j}{\partial x^b} \right) + \frac{\partial y^j}{\partial x^b} \frac{\partial}{\partial y^l} \left(\frac{\partial y^i}{\partial x^a} \right) \right) + \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} \frac{\partial}{\partial y^l} \left(\tilde{g}_{ij} \right) \right)$$

$$= \frac{\partial y^l}{\partial x^k} \left(\tilde{g}_{ij} \left(\frac{\partial y^i}{\partial x^a} \frac{\partial^2 y^j}{\partial y^l \partial x^b} + \frac{\partial y^j}{\partial x^b} \frac{\partial^2 y^i}{\partial y^l \partial x^a} \right) + \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} \frac{\partial \tilde{g}_{ij}}{\partial y^l} \right)$$

$$= \frac{\partial y^l}{\partial x^k} \frac{\partial y^i}{\partial x^a} \frac{\partial^2 y^j}{\partial y^l \partial x^b} \tilde{g}_{ij} + \frac{\partial y^l}{\partial x^k} \frac{\partial y^j}{\partial x^b} \frac{\partial^2 y^i}{\partial y^l \partial x^a} \tilde{g}_{ij} + \frac{\partial y^l}{\partial x^k} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} \frac{\partial \tilde{g}_{ij}}{\partial y^l}$$

Тоест:

$$\begin{aligned} \nabla g = \Bigg(& \frac{\partial y^l}{\partial x^k} \frac{\partial y^i}{\partial x^a} \frac{\partial^2 y^j}{\partial y^l \partial x^b} \tilde{g}_{ij} + \frac{\partial y^l}{\partial x^k} \frac{\partial y^j}{\partial x^b} \frac{\partial^2 y^i}{\partial y^l \partial x^a} \tilde{g}_{ij} + \frac{\partial y^l}{\partial x^k} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} \frac{\partial \tilde{g}_{ij}}{\partial y^l} \\ & - \left(\tilde{\Gamma}^n{}_{il} - \frac{\partial y^n}{\partial x^p} \frac{\partial^2 x^p}{\partial y^i \partial y^l} \right) \cancel{\frac{\partial x^m}{\partial y^n}} \frac{\partial y^i}{\partial x^a} \frac{\partial y^l}{\partial x^k} \tilde{g}_{nj} \cancel{\frac{\partial y^n}{\partial x^m}} \frac{\partial y^j}{\partial x^b} \\ & - \left(\tilde{\Gamma}^n{}_{lj} - \frac{\partial y^n}{\partial x^p} \frac{\partial^2 x^p}{\partial y^l \partial y^j} \right) \cancel{\frac{\partial x^m}{\partial y^n}} \frac{\partial y^l}{\partial x^k} \frac{\partial y^j}{\partial x^b} \tilde{g}_{ni} \cancel{\frac{\partial y^n}{\partial x^m}} \frac{\partial y^i}{\partial x^a} \\ & \Bigg) \frac{\partial x^a}{\partial y^i} \mathrm{d}y^i \frac{\partial x^b}{\partial y^j} \mathrm{d}y^j \frac{\partial x^k}{\partial y^l} \mathrm{d}y^l \end{aligned}$$

$$\tilde{g}_{nj} = \tilde{g}_{ij}\frac{\partial y^i}{\partial y^n} \implies \frac{\partial y^n}{\partial x^p}\frac{\partial^2 x^p}{\cancel{\partial y^i}\partial y^l}\frac{\cancel{\partial y^i}}{\partial x^a}\frac{\partial y^l}{\partial x^k}\tilde{g}_{nj}\frac{\partial y^j}{\partial x^b} = \frac{\cancel{\partial y^p}}{\partial x^p}\frac{\partial^2 x^p}{\partial x^a \partial y^l}\tilde{g}_{ij}\frac{\partial y^i}{\cancel{\partial y^p}}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}$$

$$= \frac{\partial y^i}{\partial x^p}\frac{\partial}{\partial x^a}\left(\frac{\partial x^p}{\partial y^l}\right)\tilde{g}_{ij}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b} = \left(\frac{\partial}{\partial x^a}\left(\frac{\partial y^i}{\cancel{\partial x^p}}\frac{\cancel{\partial x^p}}{\partial y^l}\right) - \frac{\partial x^p}{\partial y^l}\frac{\partial}{\partial x^a}\left(\frac{\partial y^i}{\partial x^p}\right)\right)\tilde{g}_{ij}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}$$

$$= 0 - \frac{\cancel{\partial x^p}}{\partial y^l}\frac{\partial^2 y^i}{\partial x^a \cancel{\partial x^p}}\tilde{g}_{ij}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b} = -\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\frac{\partial^2 y^i}{\partial y^l \partial x^a}\tilde{g}_{ij}$$

Аналогично $\frac{\partial y^n}{\partial x^p}\frac{\partial^2 x^p}{\partial y^l \partial y^j}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\tilde{g}_{ni}\frac{\partial y^i}{\partial x^a} = -\frac{\partial y^l}{\partial x^k}\frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^j}{\partial y^l \partial x^b}\tilde{g}_{ij}$

Следователно:

$$\begin{aligned}
\nabla g = \Bigg(&\cancel{\frac{\partial y^l}{\partial x^k}\frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^j}{\partial y^l \partial x^b}\tilde{g}_{ij}} + \cancel{\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\frac{\partial^2 y^i}{\partial y^l \partial x^a}\tilde{g}_{ij}} + \frac{\partial y^l}{\partial x^k}\frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\frac{\partial \tilde{g}_{ij}}{\partial y^l} \\
&- \tilde{\Gamma}^n{}_{il}\frac{\partial y^i}{\partial x^a}\frac{\partial y^l}{\partial x^k}\tilde{g}_{nj}\frac{\partial y^j}{\partial x^b} - \cancel{\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\frac{\partial^2 y^i}{\partial y^l \partial x^a}\tilde{g}_{ij}} \\
&- \tilde{\Gamma}^n{}_{lj}\frac{\partial y^l}{\partial x^k}\frac{\partial y^j}{\partial x^b}\tilde{g}_{ni}\frac{\partial y^i}{\partial x^a} - \cancel{\frac{\partial y^l}{\partial x^k}\frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^j}{\partial y^l \partial x^b}\tilde{g}_{ij}} \\
&\Bigg)\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\frac{\partial x^k}{\partial y^l}\mathrm{d}y^i\mathrm{d}y^j\mathrm{d}y^l \\
&= \left(\frac{\partial \tilde{g}_{ij}}{\partial y^l} - \tilde{\Gamma}^n{}_{il}\tilde{g}_{nj} - \tilde{\Gamma}^n{}_{lj}\tilde{g}_{ni}\right)\cancel{\frac{\partial y^l}{\partial x^k}}\frac{\partial y^i}{\cancel{\partial x^a}}\frac{\cancel{\partial y^j}}{\partial x^b}\cancel{\frac{\partial x^a}{\partial y^i}}\frac{\partial x^b}{\partial y^j}\frac{\cancel{\partial x^k}}{\partial y^l}\mathrm{d}y^i\mathrm{d}y^j\mathrm{d}y^l \\
&= \tilde{\nabla}_l\,\tilde{g}_{ij}\,\mathrm{d}y^i\mathrm{d}y^j\mathrm{d}y^l = \tilde{\nabla}\tilde{g}
\end{aligned}$$

4. Решение Правим смяна на променлива, от x към y, което означава че:

$$\Gamma^i{}_{jk} = \tilde{\Gamma}^a{}_{bc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^k} \qquad \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^a}\frac{\partial y^a}{\partial x^i} \qquad \mathrm{d}x^a = \frac{\partial x^a}{\partial y^i}\mathrm{d}y^i$$

Тоест:

$$\begin{aligned}
R^i{}_{jkl}&\frac{\partial}{\partial x^i}\mathrm{d}x^j\mathrm{d}x^k\mathrm{d}x^l = \left(\frac{\partial}{\partial x^l}\Gamma^i{}_{jk} - \frac{\partial}{\partial x^k}\Gamma^i{}_{jl} + \Gamma^i{}_{ml}\Gamma^m{}_{jk} - \Gamma^i{}_{mk}\Gamma^m{}_{jl}\right)\frac{\partial}{\partial x^i}\mathrm{d}x^j\mathrm{d}x^k\mathrm{d}x^l \\
&= \Bigg(\frac{\partial}{\partial y^d}\frac{\partial y^d}{\partial x^l}\left(\tilde{\Gamma}^a{}_{bc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^k}\right) - \frac{\partial}{\partial y^c}\frac{\partial y^c}{\partial x^k}\left(\tilde{\Gamma}^a{}_{bd}\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^l}\right) \\
&\quad + \left(\tilde{\Gamma}^a{}_{nd}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^d}{\partial x^l} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m \partial x^l}\right)\left(\tilde{\Gamma}^n{}_{bc}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k} + \frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^k}\right) \\
&\quad - \left(\tilde{\Gamma}^a{}_{nc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^c}{\partial x^k} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m \partial x^k}\right)\left(\tilde{\Gamma}^n{}_{bd}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l} + \frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^l}\right) \\
&\quad \Bigg)\frac{\partial}{\partial y^a}\frac{\partial y^a}{\partial x^i}\frac{\partial x^j}{\partial y^b}\mathrm{d}y^b\frac{\partial x^k}{\partial y^c}\mathrm{d}y^c\frac{\partial x^l}{\partial y^d}\mathrm{d}y^d \\
&= \Bigg(\frac{\partial}{\partial y^d}\tilde{\Gamma}^a{}_{bc}\left(\frac{\partial y^d}{\partial x^l}\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k}\right) + \cancel{\frac{\partial y^d}{\partial x^l}\frac{\partial}{\partial y^d}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^k}} \\
&\quad - \frac{\partial}{\partial y^c}\tilde{\Gamma}^a{}_{bd}\left(\frac{\partial y^c}{\partial x^k}\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l}\right) - \cancel{\frac{\partial y^c}{\partial x^k}\frac{\partial}{\partial y^c}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^l}} \\
&\quad + \tilde{\Gamma}^a{}_{nd}\frac{\partial x^i}{\partial y^a}\frac{\cancel{\partial y^n}}{\cancel{\partial x^m}}\frac{\partial y^d}{\partial x^l}\tilde{\Gamma}^n{}_{bc}\frac{\cancel{\partial x^m}}{\cancel{\partial y^n}}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k} + \tilde{\Gamma}^a{}_{nd}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^d}{\partial x^l}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^k} \\
&\quad + \tilde{\Gamma}^n{}_{bc}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m \partial x^l} + \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m \partial x^l}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^k}
\end{aligned}$$

$$
\begin{aligned}
&- \tilde{\Gamma}^a{}_{nc}\frac{\partial x^i}{\partial y^a}\frac{\cancel{\partial y^n}}{\cancel{\partial x^m}}\frac{\partial y^c}{\partial x^k}\tilde{\Gamma}^n{}_{bd}\frac{\cancel{\partial x^m}}{\cancel{\partial y^n}}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l} - \tilde{\Gamma}^a{}_{nc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^c}{\partial x^k}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^l} \\
&- \tilde{\Gamma}^n{}_{bd}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m \partial x^k} - \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m \partial x^k}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^l} \\
&\Big)\frac{\partial y^a}{\partial x^i}\frac{\partial x^j}{\partial y^b}\frac{\partial x^k}{\partial y^c}\frac{\partial x^l}{\partial y^d}\frac{\partial}{\partial y^a}\mathrm{d}y^b\mathrm{d}y^c\mathrm{d}y^d
\end{aligned}
$$

$$
\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m \partial x^l}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^k} = \frac{\partial x^i}{\partial x^m}\frac{\partial^2 y^p}{\partial y^p \partial y^p}\frac{\partial x^m}{\partial x^l}\frac{\partial^2 x^l}{\partial x^j \partial x^k} = 0 \quad \text{Аналогично } \frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m \partial x^k}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^l} = 0
$$

По формулата виждаме, че $\tilde{\Gamma}^a{}_{nd} = \tilde{\Gamma}^a{}_{nc}\frac{\cancel{\partial y^q}}{\cancel{\partial y^a}}\frac{\cancel{\partial y^n}}{\cancel{\partial y^n}}\frac{\partial y^c}{\partial y^d} + \frac{\partial y^a}{\partial y^p}\frac{\partial^2 y^p}{\partial y^n \partial y^d} = \tilde{\Gamma}^a{}_{nc}\frac{\partial y^c}{\partial y^d}$, следователно:

$$
\begin{aligned}
&\tilde{\Gamma}^a{}_{nd}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^d}{\partial x^l}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^k} - \tilde{\Gamma}^a{}_{nc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^c}{\partial x^k}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^l} = \\
&\tilde{\Gamma}^a{}_{nc}\frac{\partial y^c}{\cancel{\partial y^d}}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\cancel{\partial y^d}}{\partial x^l}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^k} - \tilde{\Gamma}^a{}_{nc}\frac{\partial x^i}{\partial y^a}\frac{\partial y^n}{\partial x^m}\frac{\partial y^c}{\partial x^k}\frac{\partial x^m}{\partial y^p}\frac{\partial^2 y^p}{\partial x^j \partial x^l} = 0
\end{aligned}
$$

Аналогично $\tilde{\Gamma}^n{}_{bc}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^c}{\partial x^k}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m \partial x^l} - \tilde{\Gamma}^n{}_{bd}\frac{\partial x^m}{\partial y^n}\frac{\partial y^b}{\partial x^j}\frac{\partial y^d}{\partial x^l}\frac{\partial x^i}{\partial y^p}\frac{\partial^2 y^p}{\partial x^m \partial x^k} = 0$

Следователно:

$$
\begin{aligned}
&R^i{}_{jkl}\frac{\partial}{\partial x^i}\mathrm{d}x^j\mathrm{d}x^k\mathrm{d}x^l \\
&= \left(\frac{\partial}{\partial y^d}\tilde{\Gamma}^a{}_{bc} - \frac{\partial}{\partial y^c}\tilde{\Gamma}^a{}_{bd} + \tilde{\Gamma}^a{}_{nd}\tilde{\Gamma}^n{}_{bc} - \tilde{\Gamma}^a{}_{nc}\tilde{\Gamma}^n{}_{bd}\right)\cancel{\frac{\partial y^d}{\partial x^l}}\frac{\partial x^i}{\cancel{\partial y^a}}\frac{\partial y^b}{\partial x^j}\cancel{\frac{\partial y^c}{\partial x^k}}\cancel{\frac{\partial y^a}{\partial x^i}}\frac{\partial x^j}{\cancel{\partial y^b}}\frac{\partial x^k}{\partial y^c}\cancel{\frac{\partial x^l}{\partial y^d}}\frac{\partial}{\partial y^a}\mathrm{d}y^b\mathrm{d}y^c\mathrm{d}y^d \\
&= \tilde{R}^a{}_{bcd}\frac{\partial}{\partial y^a}\mathrm{d}y^b\mathrm{d}y^c\mathrm{d}y^d
\end{aligned}
$$

Бонус задача Докажете Тъждеството на Бианки

Решение

$$
\begin{aligned}
R^i{}_{abc} + R^i{}_{bca} + R^i{}_{cab} = &\frac{\partial}{\partial x^c}\Gamma^i{}_{ab} - \frac{\partial}{\partial x^b}\Gamma^i{}_{ac} \quad + \Gamma^i{}_{mc}\Gamma^m{}_{ab} - \Gamma^i{}_{mb}\Gamma^m{}_{ac} \\
&+ \frac{\partial}{\partial x^a}\Gamma^i{}_{bc} - \frac{\partial}{\partial x^c}\Gamma^i{}_{ba} \quad + \Gamma^i{}_{ma}\Gamma^m{}_{bc} - \Gamma^i{}_{mc}\Gamma^m{}_{ba} \\
&+ \frac{\partial}{\partial x^b}\Gamma^i{}_{ca} - \frac{\partial}{\partial x^a}\Gamma^i{}_{cb} \quad + \Gamma^i{}_{mb}\Gamma^m{}_{ca} - \Gamma^i{}_{ma}\Gamma^m{}_{cb}
\end{aligned}
$$

Използвайки формулата за смяна на променливи при кривини, виждаме че:

$$
\Gamma^i{}_{ab} = \Gamma^i{}_{ba}\frac{\partial y^i}{\partial y^i}\frac{\partial y^b}{\partial y^a}\frac{\partial y^a}{\partial y^b} + \frac{\partial y^i}{\partial y^p}\frac{\partial^2 y^p}{\partial y^a \partial y^b} = \Gamma^i{}_{ba}
$$

Тоест:

$$
\begin{aligned}
&\frac{\partial}{\partial x^c}\Gamma^i{}_{ab} - \frac{\partial}{\partial x^c}\Gamma^i{}_{ba} = \frac{\partial}{\partial x^c}\Gamma^i{}_{ab} - \frac{\partial}{\partial x^c}\Gamma^i{}_{ab} = 0 \\
&\text{Аналогично } \frac{\partial}{\partial x^b}\Gamma^i{}_{ca} - \frac{\partial}{\partial x^b}\Gamma^i{}_{ac} = 0 \quad \frac{\partial}{\partial x^a}\Gamma^i{}_{bc} - \frac{\partial}{\partial x^a}\Gamma^i{}_{cb} = 0
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma^i{}_{mc}\Gamma^m{}_{ab} - \Gamma^i{}_{mc}\Gamma^m{}_{ba} = \Gamma^i{}_{mc}\Gamma^m{}_{ab} - \Gamma^i{}_{mc}\Gamma^m{}_{ab} = 0 \\
&\text{Аналогично } \Gamma^i{}_{mb}\Gamma^m{}_{ca} - \Gamma^i{}_{mb}\Gamma^m{}_{ac} = 0 \quad \Gamma^i{}_{ma}\Gamma^m{}_{bc} - \Gamma^i{}_{ma}\Gamma^m{}_{cb} = 0
\end{aligned}
$$

Следователно $R^i{}_{abc} + R^i{}_{bca} + R^i{}_{cab} = 0$

5. Решение Правим смяна на координатна система от (x^0, x^1, x^2, x^3) на $(x^0, R, \varphi, \theta)$, такава че:

$$x^1 = R\cos\theta\cos\varphi \quad x^2 = R\cos\theta\sin\varphi \quad x^3 = R\sin\theta$$

6. Решение 0MI0800028, следователно ще разгледаме отместването на перихелия на Нептун. Ще използваме, че:

- гравитационния радиус на слънцето е $\alpha_{\text{Слънце}} = 2.953$ km
- средния радиус на Нептун е $r_{\text{средно,Нептун}} = 4.5 \cdot 10^9$ km
- ексцентритетът на Нептун е $e = 0.008678$
- Нептун прави една обиколка (около Слънцето) на всеки 164.8 години

По формулата на Айнщайн:

$$\Delta\varphi \approx \frac{3\pi\alpha_{\text{Слънце}}}{r_{\text{средно,Нептун}}(1-e^2)} \approx \frac{3\pi \cdot 2.953}{4.5 \cdot 10^9 \cdot (1 - 0.008678^2)} \approx \pi\frac{8859 \cdot 10^{-3}}{4.5 \cdot 10^9 \cdot (1 - 0.000075307684)}$$
$$\approx \pi\frac{8859}{4.5 \cdot 10^{12} \cdot 0.999924692316} \approx 6.18521453154852035278818 59... \cdot 10^{-9} \text{ радиани на обиколка}$$

Превръщайки към дъгови секунди на 100 години :

$$\approx 0.000000006185214531548520352788185 9... \cdot \frac{180}{\pi}3600 \cdot \frac{100}{164.8}$$
$$\approx 0.00077414567775884732...''$$

По точната формула:

$$\Delta\varphi = T_{\text{real}} - 2\pi = \frac{2\pi}{M\left(\sqrt{1 - \frac{\alpha_{\text{Слънце}}}{r_{\min}} - \frac{2\alpha_{\text{Слънце}}}{r_{\max}}}, \sqrt{1 - \frac{2\alpha_{\text{Слънце}}}{r_{\min}} - \frac{\alpha_{\text{Слънце}}}{r_{\max}}}\right)} - 2\pi$$

$$r_{\min} = (1-e)r_{\text{средно,Нептун}} = (1 - 0.008678) \cdot 4.5 \cdot 10^9 = 0.991322 \cdot 4.5 \cdot 10^9 = 4.460949 \cdot 10^9$$
$$r_{\max} = (1+e)r_{\text{средно,Нептун}} = (1 + 0.008678) \cdot 4.5 \cdot 10^9 = 1.008678 \cdot 4.5 \cdot 10^9 = 4.539051 \cdot 10^9$$
$$\frac{\alpha_{\text{Слънце}}}{r_{\min}} = \frac{2.953}{4.460949 \cdot 10^9} = \frac{2953}{4460949 \cdot 10^3} \qquad \frac{\alpha_{\text{Слънце}}}{r_{\max}} = \frac{2.953}{4.539051 \cdot 10^9} = \frac{2953}{4539051 \cdot 10^3}$$

$$\sqrt{1 - \frac{\alpha_{\text{Слънце}}}{r_{\min}} - \frac{2\alpha_{\text{Слънце}}}{r_{\max}}} = \sqrt{1 - \frac{2953}{4460949 \cdot 10^3} - 2\frac{2953}{4539051 \cdot 10^3}}$$
$$= \sqrt{\frac{4460949 \cdot 10^3 \cdot 4539051 \cdot 10^3 - 2953 \cdot 4539051 - 2 \cdot 2953 \cdot 4460949}{4460949 \cdot 10^3 \cdot 4539051 \cdot 10^3}}$$
$$= \sqrt{\frac{1}{4460949 \cdot 10^3 \cdot 4539051 \cdot 10^3}}\sqrt{20248474979648817603}$$

$$\sqrt{1 - \frac{2\alpha_{\text{Слънце}}}{r_{\min}} - \frac{\alpha_{\text{Слънце}}}{r_{\max}}} = \sqrt{1 - 2\frac{2953}{4460949 \cdot 10^3} - \frac{2953}{4539051 \cdot 10^3}}$$
$$= \sqrt{\frac{4460949 \cdot 10^3 \cdot 4539051 \cdot 10^3 - 2 \cdot 2953 \cdot 4539051 - 2953 \cdot 4460949}{4460949 \cdot 10^3 \cdot 4539051 \cdot 10^3}}$$
$$= \sqrt{\frac{1}{4460949 \cdot 10^3 \cdot 4539051 \cdot 10^3}}\sqrt{20248474979418182397}$$

$$
\begin{aligned}
&M\left(\sqrt{1-\frac{\alpha_{\text{Слънце}}}{r_{\min}}-\frac{2\alpha_{\text{Слънце}}}{r_{\max}}},\sqrt{1-\frac{2\alpha_{\text{Слънце}}}{r_{\min}}-\frac{\alpha_{\text{Слънце}}}{r_{\max}}}\right)\\
&=M\left(\sqrt{\frac{1}{4460949\cdot10^3\cdot4539051\cdot10^3}}\sqrt{20248474979648817603}\,,\right.\\
&\qquad\left.\sqrt{\frac{1}{4460949\cdot10^3\cdot4539051\cdot10^3}}\sqrt{20248474979418182397}\right)\\
&=\sqrt{\frac{1}{4460949\cdot4539051\cdot10^6}}M\left(\sqrt{20248474979648817603},\sqrt{20248474979418182397}\right)\\
&=2.2223059021530026666489...\cdot10^{-10}\cdot4.4998305500911364873496...\cdot10^9\\
&=0.99999999901559253273566948868277025406747554 4...=M_{\text{стойност}}
\end{aligned}
$$

$$
\frac{2\pi}{M_{\text{стойност}}}=6.283185313364801017607169369843328031804903 50...
$$

$$
\begin{aligned}
\Delta\varphi&=6.283185313364801017607169369843328031804903 50...-2\pi\\
&=6.185214540681882603284322 2...\cdot10^{-9}\ \text{радиани на обиколка}\\
&\text{Превръщайки към дъгови секунди на 100 години :}\\
&=0.000000006185214540681882603284322 2...\cdot\frac{180}{\pi}3600\cdot\frac{100}{164.8}\\
&=0.00077414567890198518...''
\end{aligned}
$$

Приближението на Айнщайн е достатъчно точно за 11 цифри след запетаята (различаващите цифри са удебелени и подчертани):

$$
\begin{aligned}
\text{По Айнщайн: }&\Delta\varphi\approx0.0007741456\underline{\mathbf{775884732}}...''\\
\text{По точната: }&\Delta\varphi=0.0007741456\underline{\mathbf{890198518}}...''
\end{aligned}
$$